



Ignat, R., Nguyen, L., Slastikov, V., & Zarnescu, A. (2018). Unicité du tourbillon de Ginzburg–Landau de degré un dans la boule unité en dimension $N \geq 7$. *Comptes Rendus Mathématique*, 356(9), 922-926. <https://doi.org/10.1016/j.crma.2018.07.006>

Peer reviewed version

License (if available):
CC BY-NC-ND

Link to published version (if available):
[10.1016/j.crma.2018.07.006](https://doi.org/10.1016/j.crma.2018.07.006)

[Link to publication record in Explore Bristol Research](#)
PDF-document

This is the accepted author manuscript (AAM). The final published version (version of record) is available online via Elsevier at <https://doi.org/10.1016/j.crma.2018.07.006> . Please refer to any applicable terms of use of the publisher.

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available:
<http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/>

Uniqueness of degree-one Ginzburg-Landau vortex in the unit ball in dimensions $N \geq 7$

Radu Ignat^{*}, Luc Nguyen[†], Valeriy Slastikov[‡] and Arghir Zarnescu^{§¶}

Abstract

For $\varepsilon > 0$, we consider the Ginzburg-Landau functional for \mathbb{R}^N -valued maps defined in the unit ball $B^N \subset \mathbb{R}^N$ with the vortex boundary data x on ∂B^N . In dimensions $N \geq 7$, we prove that for every $\varepsilon > 0$, there exists a unique global minimizer u_ε of this problem; moreover, u_ε is symmetric and of the form $u_\varepsilon(x) = f_\varepsilon(|x|)\frac{x}{|x|}$ for $x \in B^N$.

Keywords: uniqueness, symmetry, minimizers, Ginzburg-Landau.

MSC: 35A02, 35B06, 35J50.

1 Introduction and main results

In this note, we consider the following Ginzburg-Landau type energy functional

$$E_\varepsilon(u) = \int_{B^N} \left[\frac{1}{2} |\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2) \right] dx,$$

where $\varepsilon > 0$, B^N is the unit ball in \mathbb{R}^N , $N \geq 2$, and the potential $W \in C^1((-\infty, 1]; \mathbb{R})$ satisfies

$$W(0) = 0, W(t) > 0 \text{ for all } t \in (-\infty, 1] \setminus \{0\}, \text{ and } W \text{ is convex.} \quad (1)$$

We investigate the global minimizers of the energy E_ε in the set

$$\mathcal{A} := \{u \in H^1(B^N; \mathbb{R}^N) : u(x) = x \text{ on } \partial B^N = \mathbb{S}^{N-1}\}.$$

^{*}Institut de Mathématiques de Toulouse & Institut Universitaire de France, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. Email: Radu.Ignat@math.univ-toulouse.fr

[†]Mathematical Institute and St Edmund Hall, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, United Kingdom. Email: luc.nguyen@maths.ox.ac.uk

[‡]School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom. Email: Valeriy.Slastikov@bristol.ac.uk

[§]IKERBASQUE, Basque Foundation for Science, Maria Diaz de Haro 3, 48013, Bilbao, Bizkaia, Spain.

[¶]BCAM, Basque Center for Applied Mathematics, Mazarredo 14, E48009 Bilbao, Bizkaia, Spain. (azarnescu@bcamath.org)

^{||}“Simion Stoilow” Institute of the Romanian Academy, 21 Calea Griviței, 010702 Bucharest, Romania.

The requirement that $u(x) = x$ on \mathbb{S}^{N-1} is sometimes referred in the literature as the vortex boundary condition.

We note that in our analysis the convexity of W needs not be strict; compare [6] where strict convexity is assumed.

The direct method in the calculus of variations yields the existence of a global minimizer u_ε of E_ε over \mathcal{A} for all range of $\varepsilon > 0$. Moreover, any minimizer u_ε belongs to $C^1(\overline{B^N}; \mathbb{R}^N)$ and satisfies $|u_\varepsilon| \leq 1$ and the system of PDEs (in the sense of distributions)

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon W'(1 - |u_\varepsilon|^2) \quad \text{in } B^N. \quad (2)$$

The goal of this note is to give a short proof of the uniqueness and symmetry of the global minimizer of E_ε in \mathcal{A} for *all* $\varepsilon > 0$ in dimensions $N \geq 7$. We prove that, in these dimensions, the global minimizer is unique and given by the unique radially symmetric critical point of E_ε defined by

$$u_\varepsilon(x) = f_\varepsilon(|x|) \frac{x}{|x|} \quad \text{for all } x \in B^N, \quad (3)$$

where the radial profile $f_\varepsilon : [0, 1] \rightarrow \mathbb{R}_+$ is the unique solution of

$$\begin{cases} -f_\varepsilon'' - \frac{N-1}{r} f_\varepsilon' + \frac{N-1}{r^2} f_\varepsilon = \frac{1}{\varepsilon^2} f_\varepsilon W'(1 - f_\varepsilon^2) & \text{for } r \in (0, 1), \\ f_\varepsilon(0) = 0, f_\varepsilon(1) = 1. \end{cases} \quad (4)$$

Moreover, $f_\varepsilon > 0$ and $f_\varepsilon' > 0$ in $(0, 1)$ (see e.g. [4]).

THEOREM 1. *Assume that W satisfies (1). If $N \geq 7$, then for every $\varepsilon > 0$, u_ε given in (3) is the unique global minimizer of E_ε in \mathcal{A} .*

To our knowledge, the question about the uniqueness of minimizers/critical points of E_ε in \mathcal{A} for any $\varepsilon > 0$ was raised in dimension $N = 2$ in the book of Bethuel, Brezis and Hélein [1, Problem 10, page 139], and in general dimensions $N \geq 2$ and also for the blow-up limiting problem around the vortex (when the domain is the whole space \mathbb{R}^N and by rescaling, ε can be assumed equal to 1) in an article of Brezis [2, Section 2].

It is well known that uniqueness is present for large enough $\varepsilon > 0$ for any $N \geq 2$. Indeed, for any $\varepsilon > (W'(1)/\lambda_1)^{1/2}$ where λ_1 is the first eigenvalue of $-\Delta$ in B^N with zero Dirichlet boundary condition, E_ε is strictly convex in \mathcal{A} and thus has a unique critical point in \mathcal{A} (that is the global minimizer of our problem).

For *sufficiently small* $\varepsilon > 0$ all results regarding uniqueness question available in the literature are in the affirmative. In particular, we have:

- (i) Pacard and Rivière [11, Theorem 10.2] showed in dimension $N = 2$ that, for small $\varepsilon > 0$, E_ε has in fact a unique critical point in \mathcal{A} .
- (ii) Mironescu [10] showed in dimension $N = 2$ that, when B^2 is replaced by \mathbb{R}^2 and $\varepsilon = 1$, a local minimizer of E_ε subjected to a degree-one boundary condition at infinity is

unique (up to translation and suitable rotation). This was generalized to dimension $N = 3$ by Millot and Pisante [9] and dimensions $N \geq 4$ by Pisante [12], also in the case of the blow-up limiting problem on \mathbb{R}^N and $\varepsilon = 1$.

These results should be compared to those for the limit problem on the unit ball obtained by sending $\varepsilon \rightarrow 0$. In this limit, the Ginzburg-Landau problem ‘converges’ to the harmonic map problem from B^N to \mathbb{S}^{N-1} . It is well known that, the vortex boundary condition gives rise to a unique minimizing harmonic map $x \mapsto \frac{x}{|x|}$ if $N \geq 3$; see Brezis, Coron and Lieb [3] in dimension $N = 3$, Jäger and Kaul [7] in dimensions $N \geq 7$, and Lin [8] in dimensions $N \geq 3$.

We highlight that, in contrast to the above, our result holds for *all* $\varepsilon > 0$, provided that $N \geq 7$. The method of our proof deviates somewhat from that in the aforementioned works. In fact it is reminiscent of our recent work [6] on the (non-)uniqueness and symmetry of minimizers of the Ginzburg-Landau functionals for \mathbb{R}^M -valued maps defined on N -dimensional domains, where M is not necessarily the same as N . However we note that the results in [6] do not directly apply to the present context, as in [6] it is required that W be *strictly convex*. Furthermore, a priori, it is not clear why non-strict convexity of the potential W is sufficient to ensure uniqueness of global minimizers.

We exploit the convexity of W to lower estimate the ‘excess’ energy by a suitable quadratic energy which can be handled by the factorization trick à la Hardy. Indeed, the positivity of the excess energy is then related to the validity of a Hardy-type inequality, which explains our restriction of $N \geq 7$. This echoes our observation made in [6] that a result of Jäger and Kaul [7] on the minimality of the equator map in these dimensions is related to a certain inequality involving the sharp constant in the Hardy inequality.

We expect that our result remains valid in dimensions $2 \leq N \leq 6$, but this goes beyond the scope of this note and remains for further investigation.

2 Proof of Theorem 1

Theorem 1 will be obtained as a consequence of a stronger result on the uniqueness of global minimizers of for the \mathbb{R}^M -valued Ginzburg-Landau functional with $M \geq N$. By a slight abuse of notation, we consider the energy functional

$$E_\varepsilon(u) = \int_{B^N} \left[\frac{1}{2} |\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2) \right] dx,$$

where u belongs to

$$\mathcal{A} := \{u \in H^1(B^N; \mathbb{R}^M) : u(x) = x \text{ on } \partial B^N = \mathbb{S}^{N-1} \subset \mathbb{R}^M\}.$$

THEOREM 2. *Assume that W satisfies (1). If $M \geq N \geq 7$, then for every $\varepsilon > 0$, u_ε given in (3) is the unique global minimizer of E_ε in \mathcal{A} .*

When W is strictly convex, the above theorem is proved in [6]; see Theorem 1.7. The argument therein uses the strict convexity in a crucial way.

Proof. The proof will be done in several steps. First, we consider the difference between the energies of the critical point u_ε , defined in (3), and an arbitrary competitor $u_\varepsilon + v$ and show that this difference is controlled from below by some quadratic energy functional $F_\varepsilon(v)$. Second, we employ the positivity of the radial profile f_ε in (4) and apply the Hardy decomposition method in order to show that $F_\varepsilon(v) \geq 0$, which proves in particular that u_ε is a global minimizer of E_ε . Finally, we characterise the situation when this difference is zero and conclude to the uniqueness of the global minimizer u_ε .

Step 1: Lower bound for energy difference. For any $v \in H_0^1(B^N; \mathbb{R}^M)$, we have

$$\begin{aligned} E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) &= \int_{B^N} \left[\nabla u_\varepsilon \cdot \nabla v + \frac{1}{2} |\nabla v|^2 \right] dx \\ &\quad + \frac{1}{2\varepsilon^2} \int_{B^N} \left[W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2) \right] dx. \end{aligned}$$

Using the convexity of W , we have

$$W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2) \geq -W'(1 - |u_\varepsilon|^2)(|u_\varepsilon + v|^2 - |u_\varepsilon|^2).$$

The last two relations imply that

$$\begin{aligned} E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) &\geq \int_{B^N} \left[\nabla u_\varepsilon \cdot \nabla v - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2) u_\varepsilon \cdot v \right] dx \\ &\quad + \int_{B^N} \left[\frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - f_\varepsilon^2) |v|^2 \right] dx. \end{aligned}$$

Moreover, by (2), we obtain

$$E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq \int_{B^N} \left[\frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - f_\varepsilon^2) |v|^2 \right] dx =: \frac{1}{2} F_\varepsilon(v) \quad (5)$$

for all $v \in H_0^1(B^N; \mathbb{R}^M)$.

Step 2: A rewriting of $F_\varepsilon(v)$ using the decomposition $v = f_\varepsilon w$ for every scalar test function $v \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R})$. We consider the operator

$$L_\varepsilon := \frac{1}{2} \nabla_{L^2} F_\varepsilon = -\Delta - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2).$$

Using the decomposition

$$v = f_\varepsilon w$$

for the scalar function $v \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R})$, we have (see e.g. [5, Lemma A.1]):

$$\begin{aligned} F_\varepsilon(v) &= \int_{B^N} L_\varepsilon v \cdot v \, dx = \int_{B^N} w^2 L_\varepsilon f_\varepsilon \cdot f_\varepsilon \, dx + \int_{B^N} f_\varepsilon^2 |\nabla w|^2 \, dx \\ &= \int_{B^N} f_\varepsilon^2 \left(|\nabla w|^2 - \frac{N-1}{r^2} w^2 \right) dx, \end{aligned}$$

because (4) yields $L_\varepsilon f_\varepsilon \cdot f_\varepsilon = -\frac{N-1}{r^2} f_\varepsilon^2$ in B^N .

Step 3: We prove that $F_\varepsilon(v) \geq 0$ for every scalar test function $v \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R})$. Within the notation $v = f_\varepsilon w$ of Step 2 with $v, w \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R})$, we use the decomposition

$$w = \varphi g$$

with $\varphi = |x|^{-\frac{N-2}{2}}$ being the first eigenfunction of the Hardy's operator $-\Delta - \frac{(N-2)^2}{4|x|^2}$ in $\mathbb{R}^N \setminus \{0\}$ and $g \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R})$. We compute

$$|\nabla w|^2 = |\nabla \varphi|^2 g^2 + |\nabla g|^2 \varphi^2 + \frac{1}{2} \nabla(\varphi^2) \cdot \nabla(g^2).$$

As $|\nabla \varphi|^2 = \frac{(N-2)^2}{4|x|^2} \varphi^2$ and φ^2 is harmonic in $B^N \setminus \{0\}$, integration by parts yields

$$\begin{aligned} F_\varepsilon(v) &= \int_{B^N} f_\varepsilon^2 \left(|\nabla g|^2 \varphi^2 + \frac{(N-2)^2}{4r^2} \varphi^2 g^2 - \frac{N-1}{r^2} \varphi^2 g^2 \right) dx - \frac{1}{2} \int_{B^N} \nabla(\varphi^2) \cdot \nabla(f_\varepsilon^2) g^2 dx \\ &\geq \int_{B^N} f_\varepsilon^2 |\nabla g|^2 \varphi^2 dx + \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{f_\varepsilon^2}{r^2} \varphi^2 g^2 dx \\ &\geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} dx \geq 0, \end{aligned} \tag{6}$$

where we have used $N \geq 7$ and $\frac{1}{2} \nabla(\varphi^2) \cdot \nabla(f_\varepsilon^2) = 2\varphi\varphi' f_\varepsilon f'_\varepsilon \leq 0$ in $B^N \setminus \{0\}$.

Step 4: We prove that $F_\varepsilon(v) \geq 0$ for every $v \in H_0^1(B^N; \mathbb{R}^M)$ meaning that u_ε is a global minimizer of E_ε over \mathcal{A} ; moreover, $F_\varepsilon(v) = 0$ if and only if $v = 0$. Let $v \in H_0^1(B^N; \mathbb{R}^M)$. As a point has zero H^1 capacity in \mathbb{R}^N , a standard density argument implies the existence of a sequence $v_k \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}^M)$ such that $v_k \rightarrow v$ in $H^1(B^N, \mathbb{R}^M)$ and a.e. in B^N . On the one hand, by definition (5) of F_ε , since $W'(1-f_\varepsilon^2) \in L^\infty$, we deduce that $F_\varepsilon(v_k) \rightarrow F_\varepsilon(v)$ as $k \rightarrow \infty$. On the other hand, by (6) and Fatou's lemma, we deduce

$$\begin{aligned} \liminf_{k \rightarrow \infty} F_\varepsilon(v_k) &\geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \liminf_{k \rightarrow \infty} \int_{B^N} \frac{v_k^2}{r^2} dx \\ &\geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} dx. \end{aligned}$$

Therefore, we conclude that

$$F_\varepsilon(v) \geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} dx \geq 0, \quad \forall v \in H_0^1(B^N; \mathbb{R}^M),$$

implying by (5) that u_ε is a minimizer of E_ε over \mathcal{A} . Moreover, $F_\varepsilon(v) = 0$ if and only if $v = 0$.

Step 5: Conclusion. We have shown that u_ε is a global minimizer. Assume that \tilde{u}_ε is another global minimizer of E_ε over \mathcal{A} . If $v := \tilde{u}_\varepsilon - u_\varepsilon$, then $v \in H_0^1(B^N; \mathbb{R}^M)$ and by Steps 1 and 4, we have that $0 = E_\varepsilon(\tilde{u}_\varepsilon) - E_\varepsilon(u_\varepsilon) \geq F_\varepsilon(v) \geq 0$, which yields $F_\varepsilon(v) = 0$. Step 4 implies that $v = 0$, i.e., $\tilde{u}_\varepsilon = u_\varepsilon$. \square

REMARK 3. Recall that in the case $M \geq N \geq 7$, Jäger and Kaul [7] proved the uniqueness of global minimizer for harmonic map problem

$$\min_{u \in \mathcal{A}_*} \int_{B^N} |\nabla u|^2 dx,$$

where $\mathcal{A}_* = \{u \in H^1(B^N; \mathbb{S}^{M-1}) : u(x) = x \text{ on } \partial B^N = \mathbb{S}^{N-1} \subset \mathbb{S}^{M-1}\}$. This can also be seen by the method above as observed in our earlier paper [6]. We give the argument here for readers' convenience: Take a perturbation $v \in H_0^1(B^N, \mathbb{R}^M)$ of the harmonic map $u_*(x) = \frac{x}{|x|}$ such that $|u_*(x) + v(x)| = 1$ a.e. in B^N . Then, by [6, Proof of Theorem 5.1],

$$\int_{B^N} [|\nabla(u_* + v)|^2 - |\nabla u_*|^2] dx = \int_{B^N} [|\nabla v|^2 - |\nabla u_*|^2 |v|^2] dx = \int_{B^N} [|\nabla v|^2 - (N-1) \frac{|v|^2}{|x|^2}] dx.$$

Using Hardy's inequality in dimension N we arrive at

$$\int_{B^N} [|\nabla(u_* + v)|^2 - |\nabla u_*|^2] dx \geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{|v|^2}{|x|^2} dx.$$

The result follows since $N \geq 7$.

Acknowledgment.

R.I. acknowledges partial support by the ANR project ANR-14-CE25-0009-01. V.S. acknowledges support by the Leverhulme grant RPG-2014-226. A.Z. was partially supported by a Grant of the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-II-RU-TE-2014-4-0657; by the Basque Government through the BERC 2014-2017 program; and by the Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa accreditation SEV-2013-0323.

References

- [1] BETHUEL, F., BREZIS, H., AND HÉLEIN, F. *Ginzburg-Landau vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston Inc., Boston, MA, 1994.
- [2] BREZIS, H. Symmetry in nonlinear PDE's. In *Differential equations: La Pietra 1996 (Florence)*, vol. 65 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 1–12.
- [3] BREZIS, H., CORON, J.-M., AND LIEB, E. H. Harmonic maps with defects. *Comm. Math. Phys.* 107, 4 (1986), 649–705.
- [4] IGNAT, R., NGUYEN, L., SLASTIKOV, V., AND ZARNESCU, A. Uniqueness results for an ODE related to a generalized Ginzburg-Landau model for liquid crystals. *SIAM J. Math. Anal.* 46, 5 (2014), 3390–3425.

- [5] IGNAT, R., NGUYEN, L., SLASTIKOV, V., AND ZARNESCU, A. Stability of the melting hedgehog in the Landau-de Gennes theory of nematic liquid crystals. *Arch. Ration. Mech. Anal.* 215, 2 (2015), 633–673.
- [6] IGNAT, R., NGUYEN, L., SLASTIKOV, V., AND ZARNESCU, A. On the uniqueness of minimisers of Ginzburg-Landau functionals. *arXiv:1708.05040* (2017).
- [7] JÄGER, W., AND KAUL, H. Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems. *J. Reine Angew. Math.* 343 (1983), 146–161.
- [8] LIN, F.-H. A remark on the map $x/|x|$. *C. R. Acad. Sci. Paris Sér. I Math.* 305, 12 (1987), 529–531.
- [9] MILLOT, V., AND PISANTE, A. Symmetry of local minimizers for the three-dimensional Ginzburg-Landau functional. *J. Eur. Math. Soc. (JEMS)* 12, 5 (2010), 1069–1096.
- [10] MIRONESCU, P. Les minimiseurs locaux pour l'équation de Ginzburg-Landau sont à symétrie radiale. *C. R. Acad. Sci. Paris Sér. I Math.* 323, 6 (1996), 593–598.
- [11] PACARD, F., AND RIVIÈRE, T. *Linear and nonlinear aspects of vortices*, vol. 39 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2000. The Ginzburg-Landau model.
- [12] PISANTE, A. Two results on the equivariant Ginzburg-Landau vortex in arbitrary dimension. *J. Funct. Anal.* 260, 3 (2011), 892–905.